

# 色散媒质中 ADI-FDTD 的 PML

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**摘 要:** 基于交替方向隐式(ADI)技术的时域有限差分法(FDTD)是一种非条件稳定的计算方法,该方法的时间步长不受 Courant 稳定条件限制,而是由数值色散误差决定。与传统的 FDTD 相比,ADI-FDTD 增大了时间步长,从而缩短了总的计算时间。该文采用递归卷积(RC)方法导出了二维情况下色散媒质中 ADI-FDTD 的完全匹配层(PML)公式。应用推导公式计算了色散土壤中目标的散射,并与色散媒质中 FDTD 结果对比,在大量减少计算时间的情况下,两者结果符合较好。

**关键词:** 时域有限差分法,交替方向隐式技术,完全匹配层,色散媒质

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## PML Implementation for ADI-FDTD in Dispersive Media

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**Abstract** Alternating Direction Implicit-Finite Difference Time Domain(ADI-FDTD) is unconditionally stable and the maximum time step size is not limited by the Courant stability condition, but rather by numerical error. Compared with the conventional FDTD method, the time step size of ADI-FDTD can be enlarged arbitrarily. In this paper 2D PML implementation is proposed for ADI-FDTD in dispersive media using recursive convolution method. ADI-FDTD formulations for dispersive media can be derived from the simplified Perfectly Matched Layer (PML). Numerical results of ADI-FDTD with PML for dispersive media are compared with FDTD. Good agreement is observed.

**Key words** Finite Difference Time Domain(FDTD), Alternating-Direction Implicit(ADI), Perfectly Matched Layer(PML), Dispersive media

### 1 引言

时域有限差分法(FDTD)是一种简单有效的时域方法<sup>[1]</sup>,已经被广泛地用于求解电磁场问题。但是, FDTD空间网格的选取必须远小于波长,而时间步长的选取又受到Courant稳定条件的限制,这就需要相当长的仿真计算时间才能得到较精确的结果。近来出现的基于交替方向隐式 (Alternating Direction Implicit, ADI) 技术的FDTD方法<sup>[2-4]</sup>,把ADI技术应用到FDTD的迭代公式中,采用Yee氏网格形成隐式FDTD表达式,该方法是非条件稳定的,时间步长不受Courant稳定条件限制,而由数值精度决定,因此在具有相同精度的基础上可以大幅缩短计算时间。PML吸收层也在ADI-FDTD的计算中得到广泛应用<sup>[5-7]</sup>。本文采用递归卷积(RC)方法<sup>[8]</sup>导出了色散媒质中ADI-FDTD的PML表示式。对于Debye型色散,进一步导出了用于迭代计算的PML表示式,其他色散媒质的PML迭代公式可由相似方法得到。最后,应用推导公式计算了色散土壤中目标的散射,并与FDTD结果进行了比较。

### 2 色散媒质中的 PML-ADI-FDTD

考虑二维TE波,导电色散媒质中的PML方程为<sup>[9]</sup>

$$\partial D_x / \partial t + \sigma E_x + \sigma_y E_x = \partial (H_{zx} + H_{zy}) / \partial y \quad (1)$$

$$\partial D_y / \partial t + \sigma E_y + \sigma_x E_y = -\partial (H_{zx} + H_{zy}) / \partial x \quad (2)$$

$$\mu(\partial H_{zx} / \partial t) + \sigma_x^* H_{zx} = -\partial E_y / \partial x \quad (3)$$

$$\mu(\partial H_{zy} / \partial t) + \sigma_y^* H_{zy} = \partial E_x / \partial y \quad (4)$$

其中

$$D_\eta(t) = \epsilon_0 \epsilon_\infty E_\eta(t) + \epsilon_0 \int_0^t E_\eta(t-\tau) \chi_\eta(\tau) d\tau \quad (5)$$

$\eta = x, y$ ,  $\epsilon_0$  为自由空间介电常数,  $\epsilon_\infty$  为频率无穷大时的相对介电常数,  $\chi_\eta(\tau)$  为吸收层中媒质电极化率。将式(5)的时间离散化  $t = n\Delta t$ ,  $D_\eta(t)$  可以表示为

$$D_\eta^n = D_\eta(n\Delta t) = \epsilon_0 \epsilon_\infty E_\eta(n\Delta t) + \epsilon_0 \int_0^{n\Delta t} E_\eta(n\Delta t - \tau) \chi_\eta(\tau) d\tau \quad (6)$$

假设所有场分量在  $\Delta t/2$  时间内为常量,式(6)可以进一步表示为

$$D_{\eta}^n = \varepsilon_0 \varepsilon_{\infty} E_{\eta}^n + \varepsilon_0 \sum_{m=0}^{2n-1} E_{\eta}^{n-m/2} \int_{m\Delta t/2}^{(m+1)\Delta t/2} \chi_{\eta}(\tau) d\tau \quad (7)$$

令  $\chi_{\eta m} = \int_{m\Delta t/2}^{(m+1)\Delta t/2} \chi_{\eta}(\tau) d\tau$ ,  $\Delta\chi_{\eta m} = \chi_{\eta m} - \chi_{\eta(m+1)}$ , 应用式(7)可得

$$D_{\eta}^{n+1/2} - D_{\eta}^n = \varepsilon_0 (\varepsilon_{\infty} + \chi_{\eta 0}) E_{\eta}^{n+1/2} - \varepsilon_0 \varepsilon_{\infty} E_{\eta}^n - \varepsilon_0 \sum_{m=0}^{2n-1} E_{\eta}^{n-m/2} \Delta\chi_{\eta m} \quad (8)$$

将 RC 方法所得式(8)和 ADI 方法用于离散化的色散媒质 PML 方程中, 则色散媒质中 ADI-FDTD 的 PML 迭代过程可表示为:

(1) 第  $n$  步到第  $n+1/2$  步:

$$\begin{aligned} E_x^{n+1/2}(i+1/2, j) &= C_{y0} E_x^n(i+1/2, j) + C_{y1} \psi_x^n(i+1/2, j) \\ &+ C_{y1} [H_{zx}^n(i+1/2, j+1/2) + H_{zy}^n(i+1/2, j+1/2) \\ &- H_{zx}^n(i+1/2, j-1/2) - H_{zy}^n(i+1/2, j-1/2)] / \Delta y \quad (9) \end{aligned}$$

$$\begin{aligned} E_y^{n+1/2}(i, j+1/2) &= C_{x0} E_y^n(i, j+1/2) + C_{x1} \psi_y^n(i, j+1/2) \\ &- C_{x1} [H_{zx}^{n+1/2}(i+1/2, j+1/2) + H_{zy}^{n+1/2}(i+1/2, j+1/2) \\ &- H_{zx}^{n+1/2}(i-1/2, j+1/2) - H_{zy}^{n+1/2}(i-1/2, j+1/2)] / \Delta x \quad (10) \end{aligned}$$

$$\begin{aligned} H_{zx}^{n+1/2}(i+1/2, j+1/2) &= D_{x0} H_{zx}^n(i+1/2, j+1/2) \\ &- D_{x1} \cdot [E_y^{n+1/2}(i+1, j+1/2) - E_y^{n+1/2}(i, j+1/2)] / \Delta x \quad (11) \end{aligned}$$

$$\begin{aligned} H_{zy}^{n+1/2}(i+1/2, j+1/2) &= D_{y0} H_{zy}^n(i+1/2, j+1/2) \\ &+ D_{y1} [E_x^n(i+1/2, j+1) - E_x^n(i+1/2, j)] / \Delta y \quad (12) \end{aligned}$$

(2) 第  $n+1/2$  步到第  $n+1$  步:

$$\begin{aligned} E_x^{n+1}(i+1/2, j) &= C_{y0} E_x^{n+1/2}(i+1/2, j) + C_{y1} \psi_x^{n+1/2}(i+1/2, j) \\ &+ C_{y1} [H_{zx}^{n+1}(i+1/2, j+1/2) + H_{zy}^{n+1}(i+1/2, j+1/2) \\ &- H_{zx}^{n+1}(i+1/2, j-1/2) - H_{zy}^{n+1}(i+1/2, j-1/2)] / \Delta y \quad (13) \end{aligned}$$

$$\begin{aligned} E_y^{n+1}(i, j+1/2) &= C_{x0} E_y^{n+1/2}(i, j+1/2) + C_{x1} \psi_y^{n+1/2}(i, j+1/2) \\ &- C_{x1} [H_{zx}^{n+1/2}(i+1/2, j+1/2) + H_{zy}^{n+1/2}(i+1/2, j+1/2) \\ &- H_{zx}^{n+1/2}(i-1/2, j+1/2) - H_{zy}^{n+1/2}(i-1/2, j+1/2)] / \Delta x \quad (14) \end{aligned}$$

$$\begin{aligned} H_{zx}^{n+1}(i+1/2, j+1/2) &= D_{x0} H_{zx}^{n+1/2}(i+1/2, j+1/2) \\ &- D_{x1} \cdot [E_y^{n+1/2}(i+1, j+1/2) - E_y^{n+1/2}(i, j+1/2)] / \Delta x \quad (15) \end{aligned}$$

$$\begin{aligned} H_{zy}^{n+1}(i+1/2, j+1/2) &= D_{y0} H_{zy}^{n+1/2}(i+1/2, j+1/2) \\ &+ D_{y1} [E_x^{n+1}(i+1/2, j+1) - E_x^{n+1}(i+1/2, j)] / \Delta y \quad (16) \end{aligned}$$

其中

$$C_{\eta 0} = \frac{(2\varepsilon_0 \varepsilon_{\infty}) / \Delta t - (\sigma + \sigma_{\eta}) / 2}{[2\varepsilon_0 (\varepsilon_{\infty} + \chi_{\eta 0})] / \Delta t + [\sigma + \sigma_{\eta}] / 2} \quad (17)$$

$$C_{\eta 1} = \frac{1}{[2\varepsilon_0 (\varepsilon_{\infty} + \chi_{\eta 0})] / \Delta t + (\sigma + \sigma_{\eta}) / 2} \quad (18)$$

$$D_{\eta 0} = \frac{1 - (\sigma_{\eta}^* \Delta t) / (4\mu)}{1 + (\sigma_{\eta}^* \Delta t) / (4\mu)} \quad (19)$$

$$D_{\eta 1} = \frac{\Delta t / (2\mu)}{1 + (\sigma_{\eta}^* \Delta t) / (4\mu)} \quad (20)$$

$\eta = x, y$ 。对于不同的色散类型,  $\psi_{\eta}^n$ ,  $\psi_{\eta}^{n+1/2}$  和  $\chi_{\eta 0}$  具有不同的表示式。对于 Debye 色散, 频域相对介电常数  $\varepsilon(\omega) = \varepsilon_{\infty} + (\varepsilon_s - \varepsilon_{\infty}) / (1 + j\omega t_0)$ , 则相对应的 PML 吸收层相对介电常数<sup>[9]</sup>为  $\varepsilon(\omega) = \varepsilon_{\infty} + [(\varepsilon_s - \varepsilon_{\infty}) / (1 + j\omega t_0)] \cdot (1 - (\sigma_{\eta} t_0) / (\varepsilon_0 \varepsilon_s))$ ,  $\sigma_{\eta}$  和  $\sigma_{\eta}^*$  满足以下关系  $\sigma_{\eta} / (\varepsilon_0 \varepsilon_s) = \sigma_{\eta}^* / \mu$ , 其中  $\varepsilon_s$  为静态相对介电常数,  $t_0$  为迟豫时间常数。由以上关系可以得到

$$\begin{aligned} \psi_x^n(i+1/2, j) &= e^{-\Delta t / 2t_0} \psi_x^{n-1/2}(i+1/2, j) \\ &+ \frac{2\varepsilon_0}{\Delta t} (\varepsilon_s - \varepsilon_{\infty}) \left( 1 - \frac{\sigma_y t_0}{\varepsilon_0 \varepsilon_s} \right) (1 - e^{-\Delta t / 2t_0}) \cdot E_x^n(i+1/2, j) \quad (21) \end{aligned}$$

$$\begin{aligned} \psi_y^n(i, j+1/2) &= e^{-\Delta t / 2t_0} \psi_y^{n-1/2}(i, j+1/2) \\ &+ \frac{2\varepsilon_0}{\Delta t} (\varepsilon_s - \varepsilon_{\infty}) \left( 1 - \frac{\sigma_x t_0}{\varepsilon_0 \varepsilon_s} \right) (1 - e^{-\Delta t / 2t_0}) \cdot E_y^n(i, j+1/2) \quad (22) \end{aligned}$$

$$\chi_{\eta 0} = (\varepsilon_s - \varepsilon_{\infty}) \left( 1 - \frac{\sigma_{\eta} t_0}{\varepsilon_0 \varepsilon_s} \right) (1 - e^{-\Delta t / 2t_0}) \quad (23)$$

$\psi_x^{n+1/2}$ ,  $\psi_y^{n+1/2}$  具有与  $\psi_x^n, \psi_y^n$  相似的表示式。

式(9)~式(12)和式(13)~式(16)是隐式, 还需进一步推导迭代公式。将式(10)与式(11)联立消去  $H_z^{n+1/2}$ , 可以得到

$$\begin{aligned} E_y^{n+1/2}(i-1, j+1/2) &- \left( \frac{\Delta x^2}{C_{x1} D_{x1}} + \frac{D_{x1}}{D_{(x-1)1}} + 1 \right) E_y^{n+1/2}(i, j+1/2) \\ &+ \frac{D_{x1}}{D_{(x-1)1}} E_y^{n+1/2}(i+1, j+1/2) \\ &= - \frac{C_{x0} \Delta x^2}{C_{x1} D_{(x-1)1}} E_y^n(i, j+1/2) - \frac{\Delta x^2}{D_{(x-1)1}} \psi_y^n(i, j+1/2) \\ &+ \frac{\Delta x}{D_{(x-1)1}} [D_{x0} H_{zx}^n(i+1/2, j+1/2) + D_{y0} H_{zy}^n(i+1/2, j+1/2) \\ &- D_{(x-1)0} H_{zx}^n(i-1/2, j+1/2) - D_{y0} H_{zy}^n(i-1/2, j+1/2)] \\ &+ \frac{D_{y1} \Delta x}{D_{(x-1)1} \Delta y} [E_x^n(i+1/2, j+1) - E_x^n(i+1/2, j) \\ &- E_x^n(i-1/2, j+1) + E_x^n(i-1/2, j)] \quad (24) \end{aligned}$$

将式(13)与式(16)联立消去  $H_z^{n+1}$ , 可以得到

$$\begin{aligned}
 & E_x^{n+1}(i+1/2, j-1) - \left( \frac{\Delta y^2}{C_{y1} D_{(y-1)1}} + \frac{D_{y1}}{D_{(y-1)1}} + 1 \right) E_x^{n+1}(i+1/2, j) \\
 & + \frac{D_{y1}}{D_{(y-1)1}} E_x^{n+1}(i+1/2, j+1) \\
 = & - \frac{C_{y0} \Delta y^2}{C_{y1} D_{(y-1)1}} E_x^{n+1/2}(i+1/2, j) - \frac{\Delta y^2}{D_{(y-1)1}} \psi_x^{n+1/2}(i+1/2, j) \\
 & - \frac{\Delta y}{D_{(y-1)1}} [D_{x0} H_{zx}^{n+1/2}(i+1/2, j+1/2) \\
 & + D_{y0} H_{zy}^{n+1/2}(i+1/2, j+1/2) - D_{x0} H_{zx}^{n+1/2}(i+1/2, j-1/2) \\
 & - D_{y0} H_{zy}^{n+1/2}(i+1/2, j-1/2)] \\
 & + \frac{D_{x1} \Delta y}{D_{(y-1)1} \Delta x} [E_y^{n+1/2}(i+1, j+1/2) - E_y^{n+1/2}(i, j+1/2) \\
 & - E_y^{n+1/2}(i+1, j-1/2) + E_y^{n+1/2}(i, j-1/2)] \quad (25)
 \end{aligned}$$

将式(24)和式(25)分别代替式(10)和式(13), PML 吸收层中一个循环迭代过程为 9 → 24 → 11 → 12 → 25 → 14 → 15 → 16。色散媒质中 ADI-FDTD 迭代方程可由  $\sigma_\eta = 0$  和  $\sigma_\eta^* = 0$  得到。式(24)和式(25)可以分别组成实三对角线性方程, 采用追赶法即可求解。

### 3 计算实例

根据以上公式, 计算了土壤中良导体圆柱的散射。计算空间尺寸 3m × 2m, 外边界采用 PML 截断, 土壤-空气界面位于 X=0.5m 处, 良导体圆柱半径 0.5m, 中心距离地面 1.5m。土壤参数  $\epsilon_s = 5.9659$ ,  $\epsilon_\infty = 5.8617$ ,  $\mu_r = 1.0$ ,  $t_0 = 2.5688 \times 10^{-10}$  s,  $\sigma = 0.016$ 。点 A 位于(0.4m, 1.0m)。

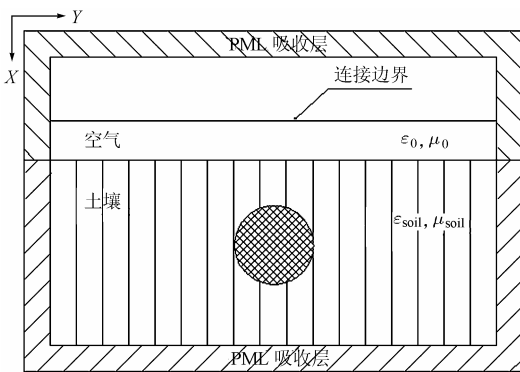


图1 土壤中良导体圆柱的散射结构

计算空间采用均匀网格剖分, 网格尺寸 0.01 m × 0.01 m, Courant 稳定条件  $\Delta t \leq 23.6$  ps。分别计算了  $f=300$  MHz 的正弦连续波和截止频率为 300 MHz 高斯脉冲激励的散射。在 X=0.45 平面加入激励源激励  $E_y$ , 应用 FDTD 和 ADI-FDTD 计算了土壤中良导体圆柱的散射, 点 A 处的电场  $E_y$  如图 2 和图 3 所示。ADI-FDTD 的 PML 吸收效果如图 4 所示, PML 电导率取

$$\sigma_\eta(\eta) = \frac{\sigma_{\eta\max} |\eta - \eta_0|^4}{d^4}, \quad \eta = x, y, \text{ 其中 } \sigma_{\eta\max} = \frac{5\epsilon_s}{4\pi}, \quad d \text{ 为 PML}$$

层厚度,  $\eta_0$  为 PML 层与色散媒质的界面。

以上仿真均采用 P III 800PC, 对于 FDTD, 取  $\Delta t = 16.67$  ps, 计算 3200 个时间步长, 耗时 303s, 所需存储量为 3.85 MByte。对于 ADI-FDTD, 取  $\Delta t = 133$  ps, 计算 400 个时间步长, 耗时 74s, 所需存储量为 4.36 MByte。ADI-FDTD 与 FDTD 结果基本相同, 而 ADI-FDTD 的计算时间仅为 FDTD 的 24%。在精度允许的范围, 适当增大 ADI-FDTD 的时间步长, 可以进一步减少计算时间。

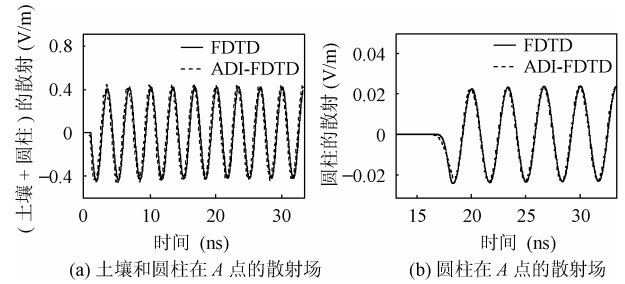


图2 正弦波激励时的散射

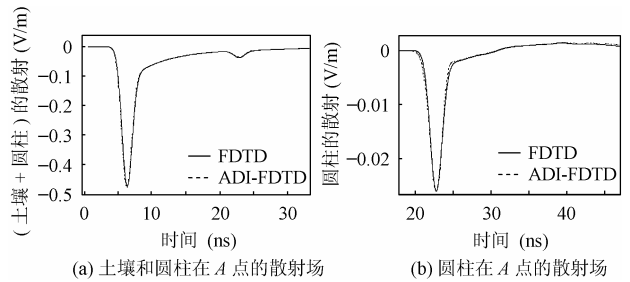


图3 高斯脉冲激励时的散射

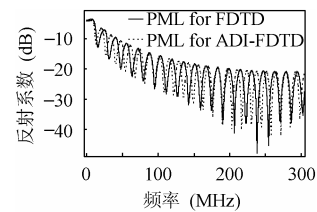


图4  $d=0.05$  m 时 PML 吸收效果

### 4 结束语

本文应用递归卷积方法导出了色散媒质中 ADI-FDTD 的 PML 吸收层, 将具有 PML 吸收层的色散媒质中 ADI-FDTD 方法用于计算土壤中目标的散射, 实例表明, ADI-FDTD 的 PML 吸收层具有与色散媒质 FDTD 的 PML 相同的吸收效果。计算结果与传统 FDTD 方法基本相同时, ADI-FDTD 可大幅缩短计算时间。该方法还可推广到其他色散媒质以及三维问题中, 使 PML-ADI-FDTD 的适应范围更为广阔。

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