双介质加载波导截止频率的边界元解法* **

吴万春　文舸一
（西北电讯工程学院）

提　　要

本文从二维亥姆霍兹方程的基本解出发，讨论了用边界元法求解双介质加载金属波导截止频率的一般原理．利用本文给出的计算公式对两个具体实例进行了计算．结果表明，该方法简单实用，并具有较高的精度．

一、引　　言

边界元法是近年来在古典边界积分法和有限元法的基础上发展起来的一种新的数值方法．它的主要特点是像边界积分法那样将原问题降低一维，即在原问题定义域的边界上去研究问题；其次它采用了有限元法中“元”的概念和插值思想．正是这些特点，使得边界元法区别于以往任何其它数值方法．边界元法早期在力学、声学等方面都得到广泛应用，近年来，已逐渐应用到电磁场与微波技术等学科中[1-4]．

本文将系统地讨论和研究边界元法用于双介质加载波导截止频率的计算．早在1950年，J. Van Bladel 在它的博士论文中讨论了双介质加载矩形波导截止频率的计算题[5]．他采用模式匹配法推得一个以截止频率作为未知数的超越方程，然后用数值方法求解该超越方程，即可得到截止频率的近似值．

一般来说，对任意形状双介质加载波导要严格求出其中模式的截止频率解析解是非常困难的．因此寻求各种计算任意形状双介质加载波导截止频率的数值方法无论在理论上还是在实践中都有一定的意义．

二、边界积分方程

如图1所示的双介质加载波导由区域 I 和区域 II 组成．

在此，假定两种介质都是无耗的，其介电常数分别为 $\varepsilon^{(I)}$ 和 $\varepsilon^{(II)}$．

* 1986年5月10日收到，1987年3月28日修改定稿．
** 中国科学院科学基金资助课题．

在区域 $i(i=\mathrm{I},\mathrm{II})$ 中，电场横向分量 $\boldsymbol{E}_t^{(i)}$ 满足亥姆霍兹方程

$$(\nabla_t^2+\omega^2\mu\varepsilon^{(i)}+\gamma^2)\boldsymbol{E}_t^{(i)}=0. \tag{1}$$

在波导壁上有

$$\boldsymbol{n}\times\boldsymbol{E}_t^{(i)}=0, \tag{2}$$

式中，$\nabla_t=\boldsymbol{i}\dfrac{\partial}{\partial x}+\boldsymbol{j}\dfrac{\partial}{\partial y}$；$\boldsymbol{n}$ 是区域 i 的边界 $\Gamma(i)=\Gamma(i,i)+\Gamma(\mathrm{I},\mathrm{II})$ 上的外法向单位矢量，$\Gamma(i,i)$ 是区域 i 的自边界，$\Gamma(\mathrm{I},\mathrm{II})$ 是区域 I 和区域 II 的公共边界．令

$$K_{ci}=\omega^2\mu\varepsilon^{(i)}+\gamma^2, \tag{3}$$

则 (1) 式可写为

$$(\nabla_t^2+K_{ci}^2)\boldsymbol{E}_t^{(i)}=0. \tag{4}$$

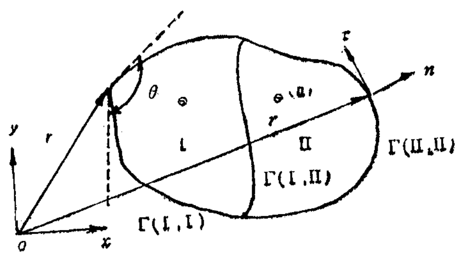

图 1 双介质加载波导

在二维矢量格林恒等式

$$\int_\Omega(\boldsymbol{Q}\cdot\nabla\times\nabla\times\boldsymbol{P}-\boldsymbol{P}\cdot\nabla\times\nabla\times\boldsymbol{Q})d\Omega=\oint_\Gamma(\boldsymbol{Q}\times\nabla\times\boldsymbol{P}-\boldsymbol{P}\times\nabla\times\boldsymbol{Q})\cdot\boldsymbol{n}d\Gamma \tag{5}$$

中，令 $\boldsymbol{Q}=\boldsymbol{E}_t^{(i)}$，$\boldsymbol{P}=G^{(i)}\boldsymbol{a}$（$\boldsymbol{a}$ 是单位常矢），同时 $G^{(i)}$ 满足

$$(\nabla_t^2+K_{ci}^2)G^{(i)}=-\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{6}$$

于是不难推得

$$C\boldsymbol{E}_t^{(i)}(\boldsymbol{r})=-\oint_{\Gamma(i)}\{[\boldsymbol{n}(\boldsymbol{r}')\times\boldsymbol{E}_t^{(i)}(\boldsymbol{r}')]\times\nabla_t'G^{(i)}+[\boldsymbol{n}(\boldsymbol{r}')\times\nabla_t'\times\boldsymbol{E}_t^{(i)}]G^{(i)}+[\boldsymbol{n}(\boldsymbol{r}')\cdot\boldsymbol{E}_t^{(i)}(\boldsymbol{r}')]\nabla_t'G^{(i)}-\boldsymbol{n}(\boldsymbol{r}')[G^{(i)}\nabla_t'\cdot\boldsymbol{E}_t^{(i)}(\boldsymbol{r}')]\}d\Gamma(\boldsymbol{r}') \tag{7}$$

式中，$C=\theta/2\pi$，θ 是观察点 $\boldsymbol{r}$ 处两条半切线的夹角，如图 1 所示．因为

$$\boldsymbol{n}(\boldsymbol{r}')=\boldsymbol{i}\cos\alpha(\boldsymbol{r}')+\boldsymbol{j}\sin\alpha(\boldsymbol{r}'), \tag{8}$$

式中 $\alpha(\boldsymbol{r}')$ 是源点上法线 $\boldsymbol{n}$ 与 x 轴的夹角，故有

$$[\boldsymbol{n}(\boldsymbol{r}')\times\boldsymbol{E}_t^{(i)}]\times\nabla_t'G^{(i)}=\boldsymbol{j}E_\tau^{(i)}\frac{\partial G^{(i)}}{\partial x'}-\boldsymbol{i}E_\tau^{(i)}\frac{\partial G^{(i)}}{\partial y'}, \tag{9}$$

$$[\boldsymbol{n}(\boldsymbol{r}')\times\nabla_t'\times\boldsymbol{E}_t^{(i)}]G^{(i)}=\{\boldsymbol{j}j\omega\mu\cos\alpha(\boldsymbol{r}')H_z^{(i)}(\boldsymbol{r}')-\boldsymbol{i}j\omega\mu\sin\alpha(\boldsymbol{r}')H_z^{(i)}(\boldsymbol{r}')\}G^{(i)}, \tag{10}$$

$$[\boldsymbol{n}(\boldsymbol{r}')\cdot\boldsymbol{E}_t^{(i)}]\nabla_t'G^{(i)}=\boldsymbol{i}E_n^{(i)}\frac{\partial G^{(i)}}{\partial x'}+\boldsymbol{j}E_n^{(i)}\frac{\partial G^{(i)}}{\partial y'}, \tag{11}$$

$$\boldsymbol{n}(\boldsymbol{r}')(G^{(i)}\nabla_t'\cdot\boldsymbol{E}_t^{(i)})=\boldsymbol{i}\gamma G^{(i)}E_z^{(i)}\cos\alpha(r')+\boldsymbol{j}\gamma G^{(i)}E_z^{(i)}\sin\alpha(\boldsymbol{r}'), \tag{12}$$

式中，下标 n 和 τ 分别表示场在边界 $\Gamma(i)$ 上的法向分量和切向分量．

将 (9)—(12) 式代入 (7) 式，并考虑到

$$\begin{cases}E_x^{(i)}=E_n^{(i)}\cos\alpha(\boldsymbol{r})-E_\tau^{(i)}\sin\alpha(\boldsymbol{r}),\\E_y^{(i)}=E_n^{(i)}\sin\alpha(\boldsymbol{r})+E_\tau^{(i)}\cos\alpha(\boldsymbol{r}),\end{cases} \tag{13}$$

立即得到

$$CE_n^{(i)}\cos\alpha(\boldsymbol{r})-CE_\tau^{(i)}\sin\alpha(\boldsymbol{r})=-\oint_{\Gamma(i)}\left\{E_n^{(i)}(\boldsymbol{r}')\frac{\partial G^{(i)}}{\partial x'}-E_\tau^{(i)}(\boldsymbol{r}')\frac{\partial G^{(i)}}{\partial y'}-\gamma E_z^{(i)}(\boldsymbol{r}')G^{(i)}\cos\alpha(\boldsymbol{r}')-j\omega\mu H_z^{(i)}(\boldsymbol{r}')\sin\alpha(\boldsymbol{r}')G^{(i)}\right\}d\Gamma(\boldsymbol{r}'), \tag{14}$$

$$CE_n^{(i)}\sin\alpha(\boldsymbol{r}) + CE_\tau^{(i)}\cos\alpha(\boldsymbol{r}) = -\oint_{\Gamma(i)}\Big\{E_n^{(i)}(\boldsymbol{r}')\frac{\partial G^{(i)}}{\partial y'} + E_\tau(\boldsymbol{r}')\frac{\partial G^{(i)}}{\partial x'}$$

$$-\gamma E_z^{(i)}(\boldsymbol{r}')G^{(i)}\sin\alpha(\boldsymbol{r}') + j\omega\mu H_z^{(i)}(\boldsymbol{r}')\cos\alpha(\boldsymbol{r}')\ G^{(i)}\Big\}d\Gamma(\boldsymbol{r}'), \tag{15}$$

上式即是所求的联立边界积分方程.

三、广义本征值方程

在推导出边界积分方程 (14) 和 (15) 后，为了数值求解，须将其化成代数方程组. 根据边界元法的一般步骤，先将 $\Gamma(i)$ 分成 N 小段，在每一小段上选择 $p-1$ 个内节点 $P(l, j)(j=1, 2, \cdots, p)$. 令 $P(l, 1)$ 和 $P(l, p+1)$ 是小段 l 的两端点，则该小段上任意一点 $P(l)$ 可用拉格朗日插值多项式近似表示为

$$P(l) = \sum_{j=1}^{p+1} N_j^{(p)}(\xi)P(l, j), \quad \xi\in[0, 1],$$

式中，$N_j^{(p)}(\xi)$ 是 p 次插值基函数:

$$N_j^{(p)}(\xi_l^{(p)}) = \delta_{jl}, \ \xi_l^{(p)} = (l-1)/p, \ (l=1, 2, \cdots, p+1),$$

这里 $\delta_{jl}=1(j=l)$ 或 $0(j\neq l)$. 通常选 $p=1$, 即每一小段近似用直线替代. 这些直线段的拼接构成近似边界 $\bar{\Gamma}(i)$，它是原边界 $\Gamma(i)$ 的近似.

在边界积分方程 (14) 和 (15) 中，积分路径 $\Gamma(i)$ 用 $\bar{\Gamma}(i)$ 代替后，再将积分化成求和，得到

$$CE_n^{(i)}\cos\alpha(\boldsymbol{r}) - CE_\tau^{(i)}\sin\alpha(\boldsymbol{r}) = -\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial x'}E_n^{(i)}d\Gamma(\boldsymbol{r}')$$

$$+\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial p'}E_\tau^{(i)}d\Gamma(\boldsymbol{r}') + \gamma\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}G^{(i)}\cos\alpha(\boldsymbol{r}')E_z^{(i)}d\Gamma(\boldsymbol{r}')$$

$$+j\omega\mu\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}G^{(i)}\sin\alpha(\boldsymbol{r})H_z^{(i)}d\Gamma(\boldsymbol{r}'), \tag{16}$$

$$CE_n^{(i)}\sin\alpha(\boldsymbol{r}) + CE_\tau^{(i)}\cos\alpha(\boldsymbol{r}) = -\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial y'}E_n^{(i)}d\Gamma(\boldsymbol{r}')$$

$$-\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial x'}E_\tau^{(i)}d\Gamma(\boldsymbol{r}') + \gamma\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}G^{(i)}\sin\alpha(\boldsymbol{r}')E_z^{(i)}d\Gamma(\boldsymbol{r}')$$

$$-j\omega\mu\sum_{j=1}^{N(i)}\int_{\bar{\Gamma}_j(i)}G^{(i)}\cos\alpha(\boldsymbol{r})H_z^{(i)}d\Gamma(\boldsymbol{r}'). \tag{17}$$

式中 $N(i)$ 表示 $\Gamma(i)$ 上剖分段数，$\bar{\Gamma}_j(i)(j=1, \cdots, N(i))$ 表示各直线元素. 现假定各场量在 $\bar{\Gamma}_j(i)$ 上是常量，于是 (16) 和 (17) 式可写成

$$\sum_{j=1}^{N(i)}\left[\frac{1}{2}\delta_{lj}\cos\alpha(\boldsymbol{r}_l) + \int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial x'}d\Gamma(\boldsymbol{r}')\right]E_{nj}^{(i)} - \sum_{j=1}^{N(i)}\left[\frac{1}{2}\delta_{lj}\sin\alpha(\boldsymbol{r}_l)\right.$$

$$+\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial y'}d\Gamma(\boldsymbol{r}')\Big]E_{\tau j}^{(i)}-\gamma\sum_{j=1}^{N(i)}\Big[\int_{\bar{\Gamma}_j(i)}G^{(i)}\cos\alpha(\boldsymbol{r}')d\Gamma(\boldsymbol{r}')\Big]E_{zj}^{(i)}$$

$$-j\omega\mu\sum_{j=1}^{N(i)}\Big[\int_{\bar{\Gamma}_j(i)}G^{(i)}\sin\alpha(\boldsymbol{r}')d\Gamma(\boldsymbol{r}')\Big]H_{zj}^{(i)}=0, \tag{18}$$

$$\sum_{j=1}^{N(i)}\Big[\frac{1}{2}\delta_{lj}\sin\alpha(\boldsymbol{r}_l)+\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial y'}d\Gamma(\boldsymbol{r}')\Big]E_{nj}^{(i)}+\sum_{j=1}^{N(i)}\Big[\frac{1}{2}\delta_{lj}\cos\alpha(\boldsymbol{r}_l)$$

$$+\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial x'}d\Gamma(\boldsymbol{r}')\Big]E_{\tau j}^{(i)}-\gamma\sum_{j=1}^{N(i)}\Big[\int_{\bar{\Gamma}_j(i)}G^{(i)}\sin\alpha(\boldsymbol{r}')d\Gamma(\boldsymbol{r}')\Big]E_{zj}^{(i)}$$

$$+j\omega\mu\sum_{j=1}^{N(i)}\Big[\int_{\bar{\Gamma}_j(i)}G^{(i)}\cos\alpha(\boldsymbol{r}')d\Gamma(\boldsymbol{r}')\Big]H_{zj}^{(i)}=0. \tag{19}$$

式中 $\boldsymbol{r}_l(l=1,2,\cdots,N(i))$ 依次取为 $\bar{\Gamma}_j(i)(j=1,2,\cdots,N(i))$ 的中点. 再将(18)和(19)式写成矩阵形式,得到

$$[P^{(i)}]\boldsymbol{E}_n^{(i)}-[Q^{(i)}]\boldsymbol{E}_\tau^{(i)}-\gamma[R^{(i)}]\boldsymbol{E}_z^{(i)}-j\omega\mu[S^{(i)}]\boldsymbol{H}_z^{(i)}=0, \tag{20}$$

$$[Q^{(i)}]\boldsymbol{E}_n^{(i)}+[P^{(i)}]\boldsymbol{E}_\tau^{(i)}-\gamma[S^{(i)}]\boldsymbol{E}_z^{(i)}+j\omega\mu[R^{(i)}]\boldsymbol{H}_z^{(i)}=0. \tag{21}$$

式中

$$\boldsymbol{E}_n^{(i)}=[E_{n1}^{(i)},\ E_{n2}^{(i)},\ \cdots,\ E_{nN(i)}^{(i)}]^T,$$
$$\boldsymbol{E}_\tau^{(i)}=[E_{\tau1}^{(i)},\ E_{\tau2}^{(i)},\ \cdots,\ E_{\tau N(i)}^{(i)}]^T,$$
$$\boldsymbol{E}_z^{(i)}=[E_{z1}^{(i)},\ E_{z2}^{(i)},\ \cdots,\ E_{zN(i)}^{(i)}]^T,$$
$$\boldsymbol{H}_z^{(i)}=[H_{z1}^{(i)},\ H_{z2}^{(i)},\ \cdots,\ H_{zN(i)}^{(i)}]^T,$$
$$P_{lj}^{(i)}=\frac{1}{2}\delta_{lj}\cos\alpha(\boldsymbol{r}_l)+\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial x'}d\Gamma(\boldsymbol{r}'),$$
$$Q_{lj}^{(i)}=\frac{1}{2}\delta_{lj}\sin\alpha(\boldsymbol{r}_l)+\int_{\bar{\Gamma}_j(i)}\frac{\partial G^{(i)}}{\partial y'}d\Gamma(\boldsymbol{r}'),$$
$$R_{lj}^{(i)}=\int_{\bar{\Gamma}_j(i)}G^{(i)}\cos\alpha(\boldsymbol{r}')d\Gamma(\boldsymbol{r}'),$$
$$S_{lj}^{(i)}=\int_{\bar{\Gamma}_j(i)}G^{(i)}\sin\alpha(\boldsymbol{r}')d\Gamma(\boldsymbol{r}').$$

设 $N(i)=N(i,\ i)+N(\mathrm{I},\ \mathrm{II})$, 这里 $N(i,\ i)$ 是自边界 $\Gamma(i,\ i)$ 上的剖分段数; $N(\mathrm{I},\mathrm{II})$ 是互边界 $\Gamma(\mathrm{I},\mathrm{II})$ 上的剖分段数. 在互边界 $\Gamma(\mathrm{I},\mathrm{II})$ 上,场必须满足连接条件:

$$\varepsilon^{(\mathrm{I})}E_{nj}^{(\mathrm{I})}=-\varepsilon^{(\mathrm{II})}E_{nj}^{(\mathrm{II})}=D_{nj},\ E_{\tau j}^{(\mathrm{I})}=E_{\tau j}^{(\mathrm{II})}=E_{\tau j},$$
$$E_{zj}^{(\mathrm{I})}=E_{zj}^{(\mathrm{II})}=E_{zj},\ H_{zj}^{(\mathrm{I})}=H_{zj}^{(\mathrm{II})}=H_{zj},$$
$$(j=N(i,i)+1,N(i,i)+2,\cdots,N(i)).$$

引入记号

$$\boldsymbol{E}_n^{(i)}(i,\ i)=[E_{n1}^{(i)},\ E_{n2}^{(i)},\ \cdots,\ E_{nN(i,i)}^{(i)}]^T,$$
$$\boldsymbol{H}_z^{(i)}(i,\ i)=[H_{z1}^{(i)},\ H_{z2}^{(i)},\ \cdots,\ H_{zN(i,i)}^{(i)}]^T,$$
$$\boldsymbol{E}_\tau(\mathrm{I},\ \mathrm{II})=[E_{\tau1},\ E_{\tau2},\ \cdots,\ E_{\tau N(\mathrm{I},\mathrm{II})}]^T,$$
$$\boldsymbol{E}_z(\mathrm{I},\ \mathrm{II})=[E_{z1},\ E_{z2},\ \cdots,\ E_{zN(\mathrm{I},\mathrm{II})}]^T,$$

$$\boldsymbol{H}_z(\mathrm{I},\mathrm{II}) = [H_{z1},\ H_{z2},\ \cdots,\ H_{zN(\mathrm{I},\mathrm{II})}]^T,$$
$$\boldsymbol{D}_n(\mathrm{I},\mathrm{II}) = [D_{n1},\ D_{n2},\ \cdots,\ D_{nN(\mathrm{I},\mathrm{II})}]^T.$$

并注意到自边界 $\Gamma(i, i)$ 上电场切向分量为零，我们有

$$\boldsymbol{E}_n^{(\mathrm{I})} = \left[\boldsymbol{E}_n^{(\mathrm{I})}(\mathrm{I},\mathrm{I}),\ \frac{1}{\varepsilon^{(\mathrm{I})}}\boldsymbol{D}_n(\mathrm{I},\mathrm{II})\right]^T,$$
$$\boldsymbol{E}_\tau^{(\mathrm{I})} = [\ 0,\quad \boldsymbol{E}_\tau(\mathrm{I},\mathrm{II})]^T,$$
$$\boldsymbol{E}_z^{(\mathrm{I})} = [\ 0,\quad \boldsymbol{E}_z(\mathrm{I},\mathrm{II})]^T,$$
$$\boldsymbol{H}_z^{(\mathrm{I})} = [\boldsymbol{H}_z^{(\mathrm{I})}(\mathrm{I},\mathrm{I}),\ \boldsymbol{H}_z(\mathrm{I},\mathrm{II})]^T,$$
$$\boldsymbol{E}_n^{(\mathrm{II})} = \left[E_n^{(\mathrm{II})}(\mathrm{II},\ \mathrm{II}),\ -\frac{1}{\varepsilon^{(\mathrm{II})}}\boldsymbol{D}_n(\mathrm{I},\mathrm{II})\right]^T,$$
$$\boldsymbol{E}_\tau^{(\mathrm{II})} = [\ 0,\quad -\boldsymbol{E}_\tau(\mathrm{I},\mathrm{II})]^T,$$
$$\boldsymbol{E}_z^{(\mathrm{II})} = [\ 0,\quad \boldsymbol{E}_z(\mathrm{I},\mathrm{II})]^T,$$
$$\boldsymbol{H}_z^{(\mathrm{II})} = [\boldsymbol{H}_z^{(\mathrm{II})}(\mathrm{II},\mathrm{II}),\ \boldsymbol{H}_z(\mathrm{I},\mathrm{II})]^T.$$

再将系数矩阵分解成分块矩阵

$$[P^{(\mathrm{I})}] = [P^{(\mathrm{I})}(\mathrm{I},\mathrm{I}),\ P^{(\mathrm{I})}(\mathrm{I},\mathrm{II})],$$
$$[P^{(\mathrm{II})}] = [P^{(\mathrm{II})}(\mathrm{II},\mathrm{II}),\ P^{(\mathrm{II})}(\mathrm{I},\mathrm{II})],$$
$$[Q^{(\mathrm{I})}] = [Q^{(\mathrm{I})}(\mathrm{I},\mathrm{I}),\ Q^{(\mathrm{I})}(\mathrm{I},\mathrm{II})],$$
$$[Q^{(\mathrm{II})}] = [Q^{(\mathrm{II})}(\mathrm{II},\mathrm{II}),\ Q^{(\mathrm{II})}(\mathrm{I},\mathrm{II})],$$
$$[R^{(\mathrm{I})}] = [R^{(\mathrm{I})}(\mathrm{I},\mathrm{I}),\ R^{(\mathrm{I})}(\mathrm{I},\mathrm{II})],$$
$$[R^{(\mathrm{II})}] = [R^{(\mathrm{II})}(\mathrm{II},\mathrm{II}),\ R^{(\mathrm{II})}(\mathrm{I},\mathrm{II})],$$
$$[S^{(\mathrm{I})}] = [S^{(\mathrm{I})}(\mathrm{I},\mathrm{I}),\ S^{(\mathrm{I})}(\mathrm{I},\mathrm{II})],$$
$$[S^{(\mathrm{II})}] = [S^{(\mathrm{II})}(\mathrm{II},\mathrm{II}),\ S^{(\mathrm{II})}(\mathrm{I},\mathrm{II})].$$

于是(20)和(21)式可写成

$$[P^{(\mathrm{I})}(\mathrm{I},\mathrm{I})]\boldsymbol{E}_n^{(\mathrm{I})}(\mathrm{I},\mathrm{I}) + \frac{1}{\varepsilon^{(\mathrm{I})}}[P^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{D}_n(\mathrm{I},\mathrm{II}) - [Q^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_\tau(\mathrm{I},\mathrm{II})$$
$$- \gamma[R^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_z(\mathrm{I},\mathrm{II}) - j\omega\mu[S^{(\mathrm{I})}(\mathrm{I},\mathrm{I})]\boldsymbol{H}_z^{(\mathrm{I})}(\mathrm{I},\mathrm{I})$$
$$- j\omega\mu[S^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{H}_z(\mathrm{I},\mathrm{II}) = 0, \tag{26}$$

$$[Q^{(\mathrm{I})}(\mathrm{I},\mathrm{I})]\boldsymbol{E}_n^{(\mathrm{I})}(\mathrm{I},\mathrm{I}) + \frac{1}{\varepsilon^{(\mathrm{I})}}[Q^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{D}_n(\mathrm{I},\mathrm{II}) + [P^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_\tau(\mathrm{I},\mathrm{II})$$
$$- \gamma[S^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_z(\mathrm{I},\mathrm{II}) + j\omega\mu[R^{(\mathrm{I})}(\mathrm{I},\mathrm{I})]\boldsymbol{H}_z^{(\mathrm{I})}(\mathrm{I},\mathrm{I})$$
$$+ j\omega\mu[R^{(\mathrm{I})}(\mathrm{I},\mathrm{II})]\boldsymbol{H}_z(\mathrm{I},\mathrm{II}) = 0, \tag{27}$$

$$[P^{(\mathrm{II})}(\mathrm{II},\mathrm{II})]\boldsymbol{E}_n^{(\mathrm{II})}(\mathrm{II},\mathrm{II}) - \frac{1}{\varepsilon^{(\mathrm{II})}}[P^{(\mathrm{II})}(\mathrm{I},\mathrm{II})]\boldsymbol{D}_n(\mathrm{I},\mathrm{II}) + [\mathbf{Q}^{(\mathrm{II})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_\tau(\mathrm{I},\mathrm{II})$$
$$- \gamma[R^{(\mathrm{II})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_z(\mathrm{I},\mathrm{II}) - j\omega\mu[S^{(\mathrm{II})}(\mathrm{II},\mathrm{II})]\boldsymbol{H}_z^{(\mathrm{II})}(\mathrm{II},\mathrm{II})$$
$$- j\omega\mu[S^{(\mathrm{II})}(\mathrm{I},\mathrm{II})]\boldsymbol{H}_z(\mathrm{I},\mathrm{II}) = 0, \tag{28}$$

$$[Q^{(\mathrm{II})}(\mathrm{II},\mathrm{II})]\boldsymbol{E}_n^{(\mathrm{II})}(\mathrm{II},\mathrm{II}) - \frac{1}{\varepsilon^{(\mathrm{II})}}[Q^{(\mathrm{II})}(\mathrm{I},\mathrm{II})]\boldsymbol{D}_n(\mathrm{I},\mathrm{II}) - [P^{(\mathrm{II})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_\tau(\mathrm{I},\mathrm{II})$$
$$- \gamma[S^{(\mathrm{II})}(\mathrm{I},\mathrm{II})]\boldsymbol{E}_z(\mathrm{I},\mathrm{II}) + j\omega\mu[R^{(\mathrm{II})}(\mathrm{II},\mathrm{II})]\boldsymbol{H}_z^{(\mathrm{II})}(\mathrm{II},\mathrm{II})$$

$$+ j\omega\mu[R^{(II)}(I, II)]\boldsymbol{H}_z(I, II) = 0. \tag{29}$$

由 (26)—(29) 式所构成的联立方程组当且仅当其系数行列式为零时才有非零解，即有

$$\det\begin{bmatrix} P^{(I)}(I,I) & -S^{(I)}(I,I) & \frac{1}{\varepsilon^{(I)}}P^{(I)}(I,II) & -Q^{(I)}(I,II) & R^{(I)}(I,II) & -S^{(I)}(I,II) & 0 & 0 \\ Q^{(I)}(I,I) & R^{(I)}(I,I) & \frac{1}{\varepsilon^{(I)}}Q^{(I)}(I,II) & P^{(I)}(I,II) & S^{(I)}(I,II) & R^{(I)}(I,II) & 0 & 0 \\ 0 & 0 & -\frac{1}{\varepsilon^{(II)}}P^{(II)}(I,II) & Q^{(II)}(I,II) & R^{(II)}(I,II) & -S^{(II)}(I,II) & P^{(II)}(II,II) & -S^{(II)}(II,II) \\ 0 & 0 & -\frac{1}{\varepsilon^{(II)}}Q^{(II)}(I,II) & -P^{(II)}(I,II) & S^{(II)}(I,II) & R^{(II)}(I,II) & Q^{(II)}(II,II) & R^{(II)}(II,II) \end{bmatrix} = 0 \tag{30}$$

此式即是广义本征值方程. 在 (30) 式中令 $\gamma = 0$，则由 (30) 式求得的本征值可确定截止频率.

四、矩阵元计算

在利用 (30) 式计算截止频率时，选取 (6) 式的解为

$$G^{(i)} = \frac{1}{4j} H_0^{(2)}(K_{ci}|\boldsymbol{r} - \boldsymbol{r}'|), \tag{31}$$

式中 $H_0^{(2)}(K_{ci}|\boldsymbol{r} - \boldsymbol{r}'|)$ 是零解第二类汉克尔函数，而 $K_{ci}^2 = \omega^2\mu\varepsilon^{(i)}$. 令 (x_j, y_j) 为 $\Gamma(i)$ 上的各节点坐标，节点编号按逆时针方向依次增大. 由 (31) 式可得

$$\frac{\partial G^{(i)}}{\partial x'} = \frac{1}{4j} H_1^{(2)}(K_{ci}|\boldsymbol{r} - \boldsymbol{r}'|)K_{ci}(x - x')/|\boldsymbol{r} - \boldsymbol{r}'|,$$

$$\frac{\partial G^{(i)}}{\partial y'} = \frac{1}{4j} H_1^{(2)}(K_{ci}|\boldsymbol{r} - \boldsymbol{r}'|)K_{ci}(y - y')/|\boldsymbol{r} - \boldsymbol{r}'|.$$

在 (22)—(25) 式中，作如下坐标变换

$$\begin{cases} x' = x_s + x\cos\theta_j - y\sin\theta_j, \\ y' = y_s + x\sin\theta_j + y\cos\theta_j, \end{cases} \tag{32}$$

式中，$\cos\theta_j = x_d/L_j$，$\sin\theta_j = y_d/L_j$，$L_j = (x_d^2 + y_d^2)^{1/2}$，$x_d = x_j - x_{j+1}$，$y_d = y_j - y_{j+1}$，$x_s = \frac{1}{2}(x_j + x_{j+1})$，$y_s = \frac{1}{2}(y_j + y_{j+1})$. $x_0 = \frac{1}{2}(x_l + x_{l+1})$，$y_0 = \frac{1}{2}(y_l + y_{l+1})$. 于是不难推出 $l \neq j$ 时，

$$P_{lj}^{(i)} = \frac{L_j}{4j}\int_{-0.5}^{0.5} \frac{H_1^{(2)}(K_{ci}\sqrt{(x\cdot x_d + x_s - x_0)^2 + (x\cdot y_d + y_s - y_0)^2})}{\sqrt{(x\cdot x_d + x_s - x_0)^2 + (x\cdot y_d + y_s - y_0)^2}} \times (x_s - x_0 + x\cdot x_d)dx, \tag{33}$$

$$Q_{lj}^{(i)} = \frac{L_j}{4j}\int_{-0.5}^{0.5} \frac{H_1^{(2)}(K_{ci}\sqrt{(x\cdot x_d + x_s - x_0)^2 + (x\cdot y_d + y_s - y_0)^2})}{\sqrt{(x\cdot x_d + x_s - x_0)^2 + (x\cdot y_d + y_s - y_0)^2}} \times (y_s - y_0 + x\cdot y_d)dx, \tag{34}$$

$$R_{lj}^{(i)}=\frac{L_j}{4j}\cos\alpha(\boldsymbol{r}_0)\int_{-0.5}^{0.5}H_0^{(2)}\left(K_{ci}\sqrt{(x\cdot x_d+x_s-x_0)^2+(x\cdot y_d+y_s-y_0)^2}\right)dx, \tag{35}$$

$$S_{lj}^{(i)}=\frac{L_j}{4j}\sin\alpha(\boldsymbol{r}_0)\int_{-0.5}^{0.5}H_0^{(2)}\left(K_{ci}\sqrt{(x\cdot x_d+x_s-x_0)^2+(x\cdot y_d+y_s-y_0)^2}\right)dx. \tag{36}$$

以上各式中 $\alpha(\boldsymbol{r}_0)=\alpha(x_0, y_0)$ 是 (x_0, y_0) 点的 α 角.

当 $l=j$ 时,各矩阵元素可用下列各式计算

$$P_{jj}^{(i)}=\frac{1}{2}\cos\alpha(\boldsymbol{r}_0), \tag{37}$$

$$Q_{jj}^{(i)}=\frac{1}{2}\sin\alpha(\boldsymbol{r}_0), \tag{38}$$

$$R_{jj}^{(i)}=\frac{L_j}{4j}\cos\alpha(\boldsymbol{r}_0)\left\{H_0^{(2)}\left(\frac{1}{2}K_{ci}L_j\right)+\frac{\pi}{2}\left[H_1^{(2)}\left(\frac{1}{2}K_{ci}L_j\right)\cdot \mathbf{H}_0\left(\frac{1}{2}K_{ci}L_j\right)-H_0^{(2)}\left(\frac{1}{2}K_{ci}L_j\right)\cdot \mathbf{H}_1\left(\frac{1}{2}K_{ci}L_j\right)\right]\right\}, \tag{39}$$

$$S_{jj}^{(i)}=\frac{L_j}{4j}\sin\alpha(\boldsymbol{r}_0)\left\{H_0^{(2)}\left(\frac{1}{2}K_{ci}L_j\right)+\frac{\pi}{2}\left[H_1^{(2)}\left(\frac{1}{2}K_{ci}L_j\right)\cdot \mathbf{H}_0\left(\frac{1}{2}K_{ci}L_j\right)-H_0^{(2)}\left(\frac{1}{2}K_{ci}L_j\right)\cdot \mathbf{H}_1\left(\frac{1}{2}K_{ci}L_j\right)\right]\right\} \tag{40}$$

在上面公式的推导过程中,我们利用了关系式

$$\int_0^x H_0^{(2)}(x)dx=x\cdot H_0^{(2)}(x)+\frac{1}{2}\pi x\left[H_1^{(2)}(x)\cdot \mathbf{H}_0(x)-H_0^{(2)}(x)\cdot \mathbf{H}_1(x)\right], \tag{41}$$

式中 $\mathbf{H}_\nu(x)$ $(\nu=0, 1)$ 是斯特鲁韦 (Struve) 函数[6],其表达式为

$$\mathbf{H}_\nu(x)=\left(\frac{x}{2}\right)^{\nu+1}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{x}{2}\right)^{2k}}{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(k+\nu+\frac{3}{2}\right)}, (\nu=0, 1).$$

上述计算公式中

$$\cos\alpha(\boldsymbol{r}_0)=-(y_l-y_{l+1})/L_l, \tag{42}$$

$$\sin\alpha(\boldsymbol{r}_0)=(x_l-x_{l+1})/L_l. \tag{43}$$

而各特殊函数的计算可采用多项式逼近的方法[7]

五、计 算 实 例

利用前面所推得的计算公式，对图 2 所示的窄边介质片加载矩形波导的主模截止频率作了计算. 计算结果示于表 1 和表 2. 表中,$K_c=\omega_c\sqrt{\mu_0\varepsilon_0}$, ω_c 是主模截止频率. 参考值 K_c 是用 J. Van Bladel 所给的公式计算出的结果.

表 1　$\varepsilon_r = 4$ 的主模截止频率

节点数目 $N(i)$	计算值 K_{ca}	百分误差(%)
16	1.931	1.046
24	1.921	0.523
32	1.917	0.314
参考值 $K_{ca} = 1.911$		

表 2　$N(i) = 32$ 时主模截止频率随 ε_r 的变化

ε_r	计算值 K_{ca}	参考值 K_{ca}	百分误差(%)
2	2.539	2.531	0.316
3	2.166	2.160	0.278
4	1.917	1.911	0.314
5	1.736	1.731	0.289
6	1.598	1.593	0.314
7	1.488	1.483	0.337
8	1.398	1.393	0.359
9	1.322	1.318	0.303
10	1.258	1.254	0.319
20	0.899	0.897	0.223
50	0.573	0.571	0.350

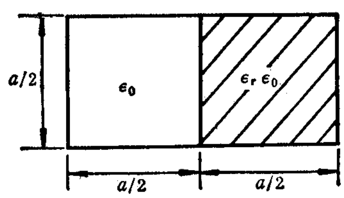

图 2　窄边介质片加载矩形波导

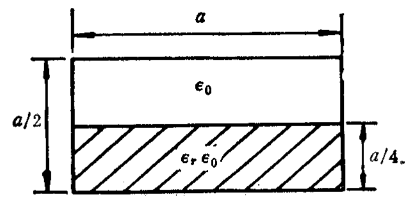

图 3　宽边介质片加载矩形波导

图 3 示出一个宽边加载矩形波导．应用前述方法的计算结果列于表 3 和表 4．

表 3　宽边加载波导的主模截止频率（$N(i) = 32$）

ε_r	计算值 K_{ca}	参考值 K_{ca}	百分误差(%)
2	2.736	2.738	0.037
10	1.962	1.963	0.051

表 4　宽边加载波导 $m = 1$ 时 LSM 模截止频率（$N(i) = 32$）

ε_r	计算值 K_{ca}	参考值 K_{ca}[8]	百分误差(%)
9	1.965	1.914	2.6

六、结 束 语

上面的理论分析和计算结果表明，用边界元法计算双介质加载波导截止频率，具有简单、实用以及精度高的特点，将本文的方法略加推广即可用于分析鳍线和槽线等结构.

参 考 文 献

[1] 吴万春、文舸一，通信学报，1986 年，第 2 期第 53—61 页.
[2] 吴万春、梁昌洪，同上，1985 年，第 3 期第 1—10 页.
[3] 吴万春等，西北电讯工程学院学报，1984 年，第 3 期，第 1—11 页.
[4] Shin Kagami, Ichiro Fukai, *IEEE Trans on MTT*, **MTT-32**(1984), 455.
[5] J. Van Bladel and T.J. Higgins, *J. Appl. Phys.*, **22**(1951)329.
[6] Milton Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, 1965.
[7] 中国科学院沈阳计算技术研究所等编，电子计算机常用算法，科学出版社，1976，第 477—480 页.
[8] 王一平等，工程电动力学，西北电讯工程出版社，1985，第 345—351 页.

THE NUMERICAL SOLUTIONS OF CUT-OFF FREQUENCY IN TWO DIELECTRIC LAYERED WAVEGUIDES BY USING BOUNDARY ELEMENT METHOD

Wu Wanchun, Wen Geyi

(Northwest Telecommunication Engineering Institute)

Based on the fundamental solution of a two dimensional Helmholtz equation, the general principle of determing the cut-off frequencies of waveguides filled with two dielectric layers by using the boundary element method is discussed. In terms of the formulae obtained, some numerical results are given for two commonly used configurations. The results show that the method is simple and practical, and its precision is appreciable.